Problems for 24/7

1. Show that the series

$$2\sin\frac{1}{3x} + 4\sin\frac{1}{9x} + \cdots + 2^n \sin\frac{1}{3^n x} + \cdots$$

converges absolutely for $x \neq 0$ but does not converge uniformly on any interval $(0, \epsilon)$ with $\epsilon > 0$.

2. Suppose that $f(t) \in C^{(n)}[0,1]$. If

$$\int_0^1 f(t)t^j \, dt = 0, \quad j = 1, 2, \ldots, n,$$

prove that $\exists \xi \in (0,1)$ such that $f^{(n)}(\xi) = 0$.

3. (a) Suppose that $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Suppose $f : \Delta \to \Delta$ is an analytic function and can be extended continuously to an arc $I \subset \partial\Delta$. Prove that if $f|_I$ is constant, then $f$ is constant on $\Delta$.

   (b) Can you prove a similar result if $D$ is a Jordan domain?

4. Let $\mathcal{L}^*$ denote Lebesgue outer measure. What are the (not necessarily measurable) sets $E$ such that $\mathcal{L}^*(E \cap [a,b]) \leq (b-a)/2$ for all intervals $[a,b]$?